# Math 254A Lecture 4 Notes

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## 1 Convexity of Set Functions and Measuring Type Classes

### 1.1 Recap + addressing superadditivity with $-\infty$

Let's fix a mistake from last time: If $a_n$ are extended reals (i.e. $\in [-\infty, \infty)$ or $(-\infty, \infty]$) and satisfy $a_{n+m} \geq a_n + a_m$ for all $n, m$, then Fekete's lemma says that $\frac{a_n}{n} \to \sup_m \frac{a_m}{m} \in (-\infty, \infty]$. However, there can be problems if $-\infty$ is allowed among $a_n$s. For example,

$$a_n = \begin{cases} 0 & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$$

does not satisfy the conclusion of Fekete's lemma. The fix is that we will need to check separately that $a_n = -\infty$ for all sufficiently large $n$.

Last time, we discussed in what situations we can turn set functions into compatible point functions. In particular, we had a topological space $X$, an open cover $\mathcal{U}$, and a map $s : \mathcal{U} \to [-\infty, \infty]$ satisfying:

(S1) If $U \subseteq U_1 \cup \cdots \cup U_k$, then $s(U) \leq \max_i s(U_i)$.

Then

$$s(x) = \inf\{s(U) : U \in \mathcal{U}, U \ni x\},$$

and $s$ is **locally finite** if $s(x) < \infty$ for all $x$. If we define

$$s(K) = \inf\{\max_i s(U_i) : K \subseteq U_1 \cup \cdots \cup U_k, U_i \in \mathcal{U}\},$$

then we had a lemma that said

$$s(K) = \sup\{s(x) : x \in K\}.$$

If we have the additional property

(S2) $s(U) = \sup\{s(K) : K \subseteq U \text{ is compact}\}$,

then we proved a lemma which says $s(U) = \sup\{s(x) : x \in U\}$.

## 1.2 Concavity of induced point functions

Now we will specialize to the situation where $X$ is a locally convex topological vector space over $\mathbb{R}$ and $\mathcal{U}$ is the collection of open, convex sets. Another lemma from last time tells us that $s : X \to \mathbb{R}$ is upper semicontinuous, i.e. for all $a \in [-\infty, \infty]$, $\{s < a\}$ is open.

**Lemma 1.1.** *Suppose a set function $s$ satisfies*

$$s\Bigg( \underbrace{\frac{1}{2}U + \frac{1}{2}V}_{=\{\frac{1}{2}u+\frac{1}{2}v : u\in U, v\in V\}} \Bigg) \geq \frac{1}{2}(s(U) + s(V)) \qquad \forall U, V \in \mathcal{U}$$

*and is locally finite. Then the point function $s$ is concave:*

$$s(tx + (1-t)y) \geq ts(x) + (1-t)s(y).$$

*Proof.* Fix $x, y$, and let $W \in \mathcal{U}$ be a neighborhood of $w := \frac{1}{2}x + \frac{1}{2}y$. Then there exist $U, V \in \mathcal{U}$ such that $U \ni x$, $V \ni y$ and $\frac{1}{2}U + \frac{1}{2}V \subseteq W$. Therefore,

$$s(W) \geq s\left(\frac{1}{2}U + \frac{1}{2}V\right) \geq \frac{1}{2}\left(s(U) + s(V)\right) \geq \frac{1}{2}(s(x) + s(y)).$$

Take the inf over $W \ni w$ to get

$$s\left(\frac{1}{2}x + \frac{1}{2}y\right) \geq \frac{1}{2}(s(x) + s(y)).$$

Now conclude that

$$s(tx + (1-t)y) \geq ts(x) + (1-t)s(y)$$

for all dyadic rational $t$ by induction on the dyadic depth of $t$. For example,

$$\begin{aligned} s\left(\frac{3}{4}x + \frac{1}{4}y\right) &= s\left(\frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{2}y\right)\right) \\ &\geq \frac{1}{2}s(x) + \frac{1}{2}s\left(\frac{1}{2}x + \frac{1}{2}y\right) \\ &\geq \frac{1}{2}s(x) + \frac{1}{2}\left(\frac{1}{2}s(x) + \frac{1}{2}s(y)\right) \\ &= \frac{3}{4}s(x) + \frac{1}{4}s(y). \end{aligned}$$

The general dyadic case is similar.

Finally, we get all $t$ by upper semicontinuity: if $t_n$ are dyadic rationals with $t_n \to t$, then

$$s(tx + (1-t)y) \geq \limsup_n s(t_n x + (1-t_n)y).$$

Now apply the previous case. $\square$

### 1.3 Measuring type classes in this setting

Here is a setting where we can apply these ideas: Let $(M, \lambda)$ be a $\sigma$-finite measure space, let $X, \mathcal{U}$ be as before, and let $\varphi : M \to X$ be a measurable map, where

- "measurable" refers to the Borel $\sigma$-algebra of $X$.
- $\varphi$ takes values inside a subset $E \subseteq X$ such that the restriction of the topology of $X$ to $E$ is separable and metrizable.

This second condition is a bit technical. Here are some examples:

**Example 1.1.** $E = X = \mathbb{R}^d$

**Example 1.2.** Let $Z$ be a compact metric space, and let $X = M(Z)$ be the collection of signed finite measures on $Z$ with the weak* topology, so $\mathcal{U}$ is the collection of weak* open convex sets. Then take $E = P(Z)$, the subset of probability measures, which is a weak*-closed convex subset of $M(Z)$ which is metrizable. In this case, we will usually have $M = Z$, $\lambda \in P(Z)$, and $\varphi$ sending $z \mapsto \delta_z$.

**Example 1.3.** Take the same as above, but $Z$ is any complete, separable metric space, and $M(Z)$ has the topology generated by all evaluations $\mu \mapsto \int f \, d\mu$ for $f \in C_b(Z)$. Still restrict $\varphi$ to take values in $P(Z)$. In this situation, $P(Z)$ still has a complete separable metric, but this is harder; we won't prove this carefully here.

Values of interest: For $U \in \mathcal{U}$, how does

$$\lambda^{\times n}\Bigg(\underbrace{\left\{ p \in M^n : \frac{1}{n}\sum_{i=1}^{n} \varphi(p_i) \in U \right\}}_{T_n(U)}\Bigg)$$

behave? Previously, we had $M = A$, $\lambda$ equals counting measure, and $\varphi(p) = \delta_p$, so $\frac{1}{n}\sum_{i=1}^{n} \varphi(p_i)$ was the empirical distribution of $p$.

**Proposition 1.1.** *There exists some $s : \mathcal{U} \to [-\infty, \infty]$ such that*

$$\lambda^{\times n}(T_n(U)) = e^{s(U)n + o(n)} \qquad \forall U \in \mathcal{U}.$$

*Proof.* Observe that if $p \in T_n(U)$ and $q \in T_m(U)$ and $r = pq$ is the concatenation, then

$$\frac{1}{n+m}\sum_{i=1}^{n+m} \varphi(r_i) = \frac{n}{n+m} \cdot \frac{1}{n}\sum_{i=1}^{n} \varphi(p_i) + \frac{m}{n+m} \cdot \frac{1}{m}\sum_{i=1}^{m} \varphi(q_i)$$

lies in $U$ if $\frac{1}{n}\sum_{i=1}^{n}\varphi(p_i)\in U$ and $\frac{1}{m}\sum_{i=1}^{m}\varphi(p_i)\in U$, i.e. $T_{n+m}(U)\supseteq T_n(U)\times T_m(U)$. So

$$\lambda^{\times(n+m)}(T_{n+m}(U))\geq\lambda^{\times n}(T_n(U))\cdot\lambda^{\times m}(T_m(U)).$$

Take logs to get superadditivity. This gives

$$\begin{aligned}s(U)&=\lim_n\underbrace{\frac{1}{n}\log\lambda^{\times n}(T_n(U))}_{a_n/n}\\&=\sup_n\frac{1}{n}\log\lambda^{\times n}(T_n(U)),\end{aligned}$$

provided that either $a_n=-\infty$ for all $n$ or $a_n\neq-\infty$ for all sufficiently large $n$. We will complete the proof next time. $\square$